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Stability of liquid–vapor flow down an inclined channel with phase change

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Abstract—The stability of a two-phase flow in a heated inclined channel is studied. The temperature of the bottom plate is below the vaporization temperature and the top plate above the vaporization temperature. A water film is on the cold wall and a vapor film on the hot wall. Gravity is the driving force. A basic flow with steady distribution of temperature but no phase change is found. The linear stability of this basic state is studied in the frame of incompressible fluid dynamics, without convection, but allowing for phase change at the interface. An ambiguity in the choice of the conditions to be required of the temperature at the phase change boundary is discussed. Three different instabilities are found, one due to the Reynolds stress when Reynolds number is large, one due to interfacial friction is associated with the viscosity difference of the two phases, and another due to phase changes at the interface.

1. INTRODUCTION

THE PROBLEM we are considering is related to the problem of stability of laminar film condensation (Nusselt's solution) on an inclined cool plate (see Unsal and Thomas [1, 2] for a fairly thorough review of this literature) and the problem of a falling film of liquid down an inclined plate which was decisively analyzed by Yih [3]. Yih's problem was generalized to two-phase Poiseuille flow between parallel plates (Blennerhassett [4], Renardy [5]).

It is necessary that we explain how our problem which is being considered here differs from those mentioned in the foregoing paragraph and allied studies. Problems of the Yih, Renardy type have fully developed basic flows but no phase change. In the present study we allow phase changes but cannot accommodate applied pressure gradients; Poiseuille flow cannot be treated rigorously in our frame. Our problem differs from the ones on laminar film condensation because the second wall and the vapor are active and the basic flow is fully developed and not of boundary layer type. The second wall allows the system to attain a steady fully developed temperature profile in the basic state which cannot exist in a semi-infinite region. The fully-developed basic flow which we study is particularly convenient for a computational study of stability since approximations are not required, and the flow could conceivably be attained in experiments. We are not in the frame of laminar film condensation because phase changes do not occur in the basic flow.

We could not obtain a steady fully developed flow for core-annular flow because there is a pressure jump across the cylindrical interface due to interfacial tension which interdicts the existence of a common saturation temperature there. For a similar reason, there are no developed steady flows with phase change which are driven by pressure gradients. On the other hand, developed steady flows in free fall between heated inclined plates can be considered if the change of density with temperature is neglected. This neglecting is usually done in the film condensation problems, and we follow this path, assuming in all that follows that the densities of the water and vapor are fixed constants independent of pressure and temperature.

2. GOVERNING EQUATIONS

We are going to assume that the density of the vapor and the density of the water are constants independent of variation of temperature or pressure across the channel. This means that the pressure will be a dynamical variable uncoupled from thermodynamics and that convective currents set up by the working of gravity on thermally induced variations of temperature will be ignored. It is conceivable that thermally induced buoyancy could produce some important effects, but Spindler [6] did allow for these effects and they were not important.

The velocities of both phases satisfy the incompressible Navier–Stokes equations:

$$\nabla \cdot \mathbf{u} = 0 \tag{1}$$

$$\rho \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = -\nabla p - \rho \mathbf{g} + \mu \nabla^2 \mathbf{u}. \tag{2}$$

In the energy equation we shall neglect the dissipation term:

$$\rho C_p \frac{\mathrm{d}T}{\mathrm{d}t} = k \nabla^2 T. \tag{3}$$

At the interface $I(\mathbf{x}, t) = y - (R - \delta(x, t)) = 0$ we have mass, momentum and energy balances:

NOMENCLATURE

B_1	energy supply due to the surface tension
B_2	interfacial friction due to the viscosity difference
B_M	mechanical energy production rate from the interface
B_T	thermal energy supply from the interface
C_p	heat capacity
c	eigenvalue
$\mathbf{D[u]}$	rate of strain
D_M	viscous dissipation in the energy of disturbances
D_T	thermal dissipation rate
$\dot{E}_M$	rate of change of mechanical energy of disturbances
$\dot{E}_T$	rate of change of thermal energy of disturbances
G	gravity contribution to the energy of disturbances
$\tilde{G}$	gravity number, $\cos\omega g H_0/V_0^2$
g	gravitational acceleration
H	half of the mean curvature of the interface
H_0	length scale
h	perturbation of the interface position
h_{fg}	latent heat, $[\![h]\!]$
$I(\mathbf{x}, t)$	interface position
I_M	mechanical energy production conversion from the basic flow
I_T	thermal energy production conversion from the basic flow
k	thermal conductivity
m	viscosity ratio, μ_2/μ_1
$\dot{m}$	mass flux through the interface
$\mathbf{n}_{12}$	normal of the interface
P	pressure of the basic flow
Pe	Peclet number, $\mathbb{R}\,Pr$
Ph	phase change contribution to the energy of disturbances
Pr	Prandtl number, $\mu c/k$
p	pressure perturbation
$\mathbb{R}$	Reynolds number, $\rho V_0 H_0/\mu$
R	width of the channel
r	dimensionless width of the channel, R/H_0
T	temperature
T_+	temperature on the hot wall
T_-	temperature on the cold wall
$\mathbf{t}$	tangent to the interface
t	time
U	velocity component along x-direction of the basic flow
$\mathbf{u}$	velocity
$\mathbf{u}_\Sigma$	interface velocity
u, v	velocity components
V_0	velocity scale
W	Weber number, $\sigma/\rho_1 H_0 V_0^2$
x, y	coordinates.

Greek symbols

α	dimensionless wave number
β	vapor volume fraction
Γ	phase change number, $k_1(T_s-T_-)h_{fg}\mu_1$
$\delta(x, t)$	thickness of the vapor layer
δ_0	thickness of the vapor layer in the basic flow
ζ	density ratio, ρ_2/ρ_1
θ	temperature perturbation
μ	dynamic viscosity
ξ	thermal conductivity ratio, k_2/k_1
Π_i	dimensionless slop of Clapeyron curve, $\rho_i T_s V_0^2/(h_{fg}(T_s-T_-))(1/\rho_1-1/\rho_2)$
ρ	density
σ	surface tension coefficient
τ_1	hot wall temperature ratio, T_+/T_s
τ_2	cold wall temperature ratio, T_-/T_s
ω	angle of inclination of the channel to the horizontal.

Indices

1	vapor
2	water
s	saturation
*	complex conjugate
~	critical value.

mass:

$$-\dot{m}(\mathbf{x}, t) = \rho_1(\mathbf{u}_1-\mathbf{u}_\Sigma)\cdot\mathbf{n}_{12} = \rho_2(\mathbf{u}_2-\mathbf{u}_\Sigma)\cdot\mathbf{n}_{12} \quad (4)$$

momentum:

$$\dot{m}[\![\mathbf{u}]\!]-[\![p]\!]\mathbf{n}_{12}+2[\![\mu\mathbf{D}[\mathbf{u}]\cdot\mathbf{n}_{12}]\!] = \nabla_\Sigma\sigma+2H\sigma\mathbf{n}_{12} \quad (5)$$

energy:

$$-[\![k\nabla T]\!]\cdot\mathbf{n}_{12}+\mathbf{u}_\Sigma\cdot(\nabla_\Sigma\sigma+2H\sigma\mathbf{n}_{12})-\frac{\partial\sigma}{\partial t}$$
$$-[\![\mathbf{u}\cdot\mathbf{T}[\mathbf{u}]\cdot\mathbf{n}_{12}]\!] = \dot{m}[\![e+\tfrac{1}{2}|\mathbf{u}|^2]\!] \quad (6)$$

with kinematic conditions:

$$\mathbf{u}_\Sigma\cdot\mathbf{n}_{12} = \frac{\partial I/\partial t}{|\nabla I|}, \quad [\![\mathbf{u}]\!]\cdot\mathbf{t} = 0 \quad (7)$$

where $\mathbf{n}_{12}$ is the normal of the interface directed from vapor to water, $\mathbf{T[u]}$ is the stress tensor and ∇_Σ is called the surface gradient. In what follows we shall take σ as constant; then, after subtracting the projection of (5) with $\mathbf{u}_\Sigma$, (6) becomes

$$-[\![k\nabla T]\!]\cdot\mathbf{n}_{12} = 2[\![\mu(\mathbf{u}-\mathbf{u}_\Sigma)\cdot\mathbf{D}[\mathbf{u}]\cdot\mathbf{n}_{12}]\!]$$
$$+\dot{m}[\![h+\tfrac{1}{2}|\mathbf{u}-\mathbf{u}_\Sigma^2|^2]\!]. \quad (8)$$

The saturation temperature at equilibrium is determined as given by the Clapeyron equation as a function of pressure

$$T = T_s = \tilde{T}(p). \tag{9}$$

Obviously, if the pressure is different on the two sides of the interface, $T_{s1} = \tilde{T}(p_{s1}) \neq \tilde{T}(p_{s2}) = T_{s2}$, the temperature cannot be the same in the water and water vapor. There will be a temperature discontinuity

$$T_1 = \tilde{T}(p_v), \quad T_2 = \tilde{T}(p_w). \tag{10}$$

This shows that thermodynamic equilibrium means that the water and its vapor are not in thermal equilibrium. The existence of a temperature discontinuity evidently cannot be eliminated by rigorous application of first principles. Schrage [7] says "...There is no reason why the temperature of the gas phase should necessarily be the same as that of the liquid or solid surface in all cases." Indeed, classic kinetic theory calculations (Pao [8], Sone and Onishi [9], Aoki and Cercignani [10], Onishi [11] and Cercignani *et al.* [12]) indicate that for monatomic vapor large temperature jumps exist at interfaces. Shankar and Deshpande [13] have measured the temperature distribution in the vapor between an evaporating liquid surface and a cooler condensing surface in water, Freon 113 and mercury. The temperature profiles obtained in mercury showed large jumps at the interface as large as almost 50% of the applied temperature difference.

The usual approximation made in the study of phase changes of liquid and vapor is to require thermal equilibrium

$$[\![T]\!] = T_1 - T_2 = 0 \tag*{(11)$_1$}$$

together with thermodynamic equilibrium in the vapor

$$T_1 = \tilde{T}(p_v). \tag*{(11)$_2$}$$

(See Plesset and Zwick [14], Gebhardt [15] and Ishii [16].) In general, $p_w \neq p_v$, so that the water is not in thermodynamic equilibrium. The choice of temperature conditions at the phase change boundary appears to be an unresolved question of physics. Two of the possibilities are expressed as (10) and (11). There are other possibilities.

The values of the parameters, which shall be taken as constants in the analysis to follow, are their table values at $T = T_s = 100°\text{C}$:

$$\begin{aligned}
&\rho_1 = 0.585\ \text{g m}^{-3};\\
&\rho_2 = 0.965\ \text{g cm}^{-3};\\
&\mu_1 = 0.0125\ \text{cP};\\
&\mu_2 = 0.28\ \text{cP};\\
&k_1 = 2.5\times10^{-4}\ \text{J cm}^{-1}\ \text{s}^{-1}\ {}^\circ\text{C}^{-1};\\
&k_2 = 6.8\times10^{-3}\ \text{J cm}^{-1}\ \text{s}^{-1}\ {}^\circ\text{C}^{-1};\\
&C_{p1} = 1.96\ \text{J g}^{-1}\ {}^\circ\text{C}^{-1};\\
&C_{p2} = 4.18\ \text{J g}^{-1}\ {}^\circ\text{C}^{-1};\\
&h_{fg} = 2.257\times10^{3}\ \text{J g}^{-1};\\
&\sigma = 64.4\ \text{dyne cm}^{-1}.
\end{aligned} \tag{12}$$

3. GOVERNING EQUATIONS FOR THE INCLINED CHANNEL

We now write the governing equations for flow down an inclined channel (Fig. 1), restricting the analysis to two space dimensions.

The interface is represented by

$$I(x, y, t) = y - (R - \delta(x, t)) = 0. \tag{13}$$

Then

$$-\mathbf{n}_{12} = \mathbf{n}_{21} = \frac{\nabla I}{|\nabla I|} = \frac{\mathbf{e}_y + \mathbf{e}_x\delta_x}{\sqrt{(1+\delta_x^2)}} \tag{14}$$

$$-\mathbf{t}_{12} = \mathbf{t}_{21} = \frac{-\mathbf{e}_y\delta_x + \mathbf{e}_x}{\sqrt{(1+\delta_x^2)}}. \tag{15}$$

The equations of motion and energy are resolved in the usual way after noting that

$$\mathbf{g} = g(\sin\omega\mathbf{e}_x - \cos\omega\mathbf{e}_y). \tag{16}$$

Then

$$\rho\frac{\mathrm{d}u}{\mathrm{d}t} = \rho g\sin\omega - \frac{\partial\phi}{\partial x} + \mu\nabla^2 u \tag{17}$$

$$\rho\frac{\mathrm{d}v}{\mathrm{d}t} = -\frac{\partial\phi}{\partial y} + \mu\nabla^2 v \tag{18}$$

where

$$\phi = p + \rho g y\cos\omega \tag{19}$$

and (3) governs T. These equations hold both in the water and vapor phase with appropriate values for the constants.

The interface conditions (4)–(11) may be expressed in coordinate form using (19) and

$$2H = \frac{\delta_{xx}}{(1+\delta_x^2)^{3/2}}. \tag{20}$$

The continuity of the tangential component of velocity $(7)_2$ implies that on $y = R - \delta$

$$[\![-v\delta_x + u]\!] = 0. \tag{21}$$

The normal and tangential component of (5) can be written, using (19), as

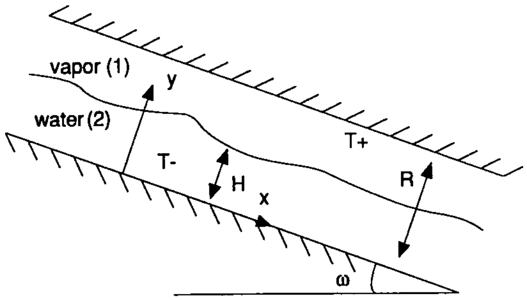

FIG. 1. Schematic of free falling flow of water (2) and vapor (1) along an inclined channel of width R. The interface is at $y = R - \delta(x, t)$.

$$-\llbracket \phi \rrbracket+\llbracket \rho \rrbracket g(R-\delta)\cos\omega-\dot{m}\frac{\llbracket v+\delta_x u\rrbracket}{(1+\delta_x^2)^{1/2}}$$
$$+\frac{2}{1+\delta_x^2}\left\llbracket \mu\left(\frac{\partial v}{\partial y}+\delta_x\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)+\delta_x^2\frac{\partial u}{\partial x}\right)\right\rrbracket$$
$$=\frac{\sigma\delta_{xx}}{(1+\delta_x^2)^{3/2}} \tag{22}$$

and

$$(1-\delta_x^2)\left\llbracket \mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right\rrbracket-2\delta_x\left\llbracket \mu\left(\frac{\partial v}{\partial y}-\frac{\partial u}{\partial x}\right)\right\rrbracket=0. \tag{23}$$

Then, since the tangential component of velocity is continuous across $y=R-\delta$

$$(\mathbf{u}-\mathbf{u}_\Sigma)=[(\mathbf{u}-\mathbf{u}_\Sigma)\cdot\mathbf{n}_{12}]\mathbf{n}_{12}=[(\mathbf{u}-\mathbf{u}_\Sigma)\cdot\mathbf{n}_{21}]\mathbf{n}_{21}$$

and

$$(\mathbf{u}-\mathbf{u}_\Sigma)\cdot\mathbf{n}_{21}=(v+\delta_x u+\delta_t)/(1+\delta_x^2)^{1/2}. \tag{24}$$

It follows that

$$\dot{m}=\rho(v+\delta_x u+\delta_t)/(1+\delta_x^2)^{1/2} \tag{25}$$

and the energy balance (8) reduces to

$$\frac{\left\llbracket k\left(-\delta_x\frac{\partial T}{\partial x}-\frac{\partial T}{\partial y}\right)\right\rrbracket}{(1+\delta_x^2)^{1/2}}=-\dot{m}\llbracket h\rrbracket-\frac{\dot{m}\llbracket(v+\delta_x u+\delta_t)^2\rrbracket}{2(1+\delta_x^2)}$$
$$+\frac{2}{(1+\delta_x^2)^{3/2}}\left\llbracket \mu(v+\delta_x u+\delta_t)\left\{\frac{\partial v}{\partial y}+\delta_x\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)+\delta_x^2\frac{\partial u}{\partial x}\right\}\right\rrbracket. \tag{26}$$

We have already mentioned that we do not know what temperature conditions to apply at the phase change boundary. We could demand that the temperature be at the saturation values in the water and water vapor. If the pressure is not continuous, then the temperature will not be continuous. We could also require that either water or its vapor be at saturation and enforce the continuity of temperature. Then the second phase will not be at saturation.

4. BASIC FLOW

There is a steady, developed, solenoidal solution

$$(u,v,T,\delta)=(U(y),0,T,\delta_0) \tag{27}$$

in the form

$$\left.\begin{aligned} U_1(y)&=-\frac{\rho_1 g\sin\omega}{2\mu_1}(R-y)^2+C_1(R-y)\\ &\quad\forall y\in(R-\delta_0,R)\\ U_2(y)&=-\frac{\rho_2 g\sin\omega}{2\mu_2}(R-y)^2+C_2(R-y)+C_3\\ &\quad\forall y\in(0,R-\delta_0)\end{aligned}\right\} \tag{28}$$

where

$$\delta_0=\frac{(T_+-T_s)k_1R}{(T_+-T_s)k_1+(T_s-T_-)k_2}\overset{\text{def}}{=}\beta R \tag{29}$$

and

$$\begin{aligned} C_1&=\frac{g\sin\omega R[\beta^2(\rho_1\mu_2-2\rho_1\mu_1+\rho_2\mu_1)+2\beta\mu_1(\rho_1-\rho_2)+\mu_1\rho_2]}{2\mu_1(\mu_1+\beta(\mu_2-\mu_1))}\\ C_2&=\frac{g\sin\omega R[\beta^2(2\rho_2\mu_2-\rho_2\mu_1-\rho_1\mu_2)-\rho_2\mu_1]}{2\mu_2(\mu_1+\beta(\mu_2-\mu_1))}\\ C_3&=\\ &\frac{g\sin\omega R^2[\rho_2\beta(\mu_2-\mu_1)-\beta^2(2\rho_2\mu_2-\rho_2\mu_1-\rho_1\mu_2)]}{2\mu_2(\mu_1+\beta(\mu_2-\mu_1))}.\end{aligned} \tag{30}$$

The temperature is given by

$$\left.\begin{aligned} T_1(y)&=T_+-(T_+-T_s)\frac{R-y}{\delta_0}\quad\forall y\in(R-\delta_0,R)\\ T_2(y)&=T_s-(T_s-T_-)\frac{R-y-\delta_0}{R-\delta_0}\quad\forall y\in(0,R-\delta_0)\end{aligned}\right\}. \tag{31}$$

At $y=R-\delta_0$, $T_1=T_2=T_s$, and the pressure

$$\left.\begin{aligned} P_1&=-\rho_1 gy\cos\omega+A_1\quad\forall y\in(R-\delta_0,R)\\ P_2&=-\rho_2 gy\cos\omega+A_2\quad\forall y\in(0,R-\delta_0)\end{aligned}\right\} \tag{32}$$

with A_1, A_2 selected so that

$$\llbracket P\rrbracket=-\llbracket\rho\rrbracket g(R-\delta_0)\cos\omega+\llbracket A\rrbracket=0 \tag{33}$$

i.e. the pressure at the vapor–water interface is continuous, and the basic flow is in thermodynamic equilibrium with

$$T_s=T(P(R-\delta_0)),\quad \dot{m}=0. \tag{34}$$

5. LINEARIZED EQUATIONS

Let u, v, p, θ, h be perturbations of U, 0, P, T, $R-\delta_0$. The linearized equations for the perturbations are

$$u_x+v_y=0 \tag{35}$$

$$u_t+Uu_x+vU'=-\frac{1}{\rho}p_x+\nu\nabla^2u \tag{36}$$

$$v_t+Uv_x=-\frac{1}{\rho}p_y+\nu\nabla^2v \tag{37}$$

$$\rho c_p(\theta_t+U\theta_x+vT')=k\nabla^2\theta. \tag{38}$$

The terms u, v and θ vanish on $y=0$ and R. The interface conditions are evaluated on $y=R-\delta_0$

$$\dot{m}=\rho_1(v_1-U_1h_x-h_t)=\rho_2(v_2-U_2h_x-h_t) \tag{39}$$

$$\llbracket u+U_yh\rrbracket=0 \tag{40}$$

$$-\llbracket p\rrbracket+\llbracket\rho\rrbracket gh\cos\omega+2\llbracket\mu v_y\rrbracket=-\sigma h_{xx} \tag{41}$$

$$[\![\mu(u_y+v_x+U_{yy}h)]\!]=0 \tag{42}$$

$$[\![k\theta_y]\!]-\dot{m}h_{fg}=0. \tag{43}$$

We have to choose a temperature condition at the phase change boundary. We require saturation in vapor or side 1

$$\theta_1=\tilde{T}'(p_s)p_1+[-T_1'+\tilde{T}'(p_s)P_1']h. \tag{44}$$

Then either

$$\theta_2=\tilde{T}'(p_s)p_2+[-T_2'+\tilde{T}'(p_s)P_2']h \tag{45}$$

or

$$[\theta]+\left[\frac{\mathrm{d}T}{\mathrm{d}y}\right]h=0. \tag{46}$$

We do not know which of the two choices (45) or (46) better represents conditions to be described at a phase change boundary.

6. PERTURBATION EQUATIONS AND NORMAL MODES

To make our equations dimensionless we use the following scales:

length: $H_0\overset{\text{def}}{=}R-\delta_0$

velocity: $V_0=\dfrac{(\rho_2+\beta(\rho_1-\rho_2))g\sin\omega}{2(\mu_1+\beta(\mu_2-\mu_1))}\delta_0H_0$

time: $\dfrac{H_0}{V_0}$

pressure: $\rho_iV_0^2$

temperature: T_s-T_-

where V_0 is just the velocity at the interface. We shall use the same letters for dimensional and dimensionless variables. If we specify the fluids as water and its vapor then all their materials parameters are determined. We are left with four independent parameters $\mathbb{R}_2$, r, τ_2, ω and $\zeta=1.65\times10^3$, $m=22.4$, $\xi=27.2$.

The basic flow in dimensionless form is given by

$$U_1(y)=a_1(r-y)^2+b_1(r-y) \tag{47}$$

$$U_2(y)=a_2(r-y)^2+b_2(r-y)+c_2 \tag{48}$$

$$T_1(y)=\frac{\tau_1}{1-\tau_2}-\frac{\tau_1-1}{1-\tau_2}\frac{r-y}{r-1} \tag{49}$$

$$T_2(y)=\frac{1}{1-\tau_2}-(1-y) \tag{50}$$

where

$$a_1=-\frac{(1+\beta(m-1))(1-\beta)}{(\zeta+\beta(1-\zeta))\beta},\quad a_2=\frac{\zeta}{m}a_1$$

$$b_1=\frac{(\beta^2(\zeta+m-2)+2\beta(1-\zeta)+\zeta)}{\beta(\zeta+\beta(1-\zeta))}$$

$$b_2=\frac{(\beta^2(2\zeta m-\zeta-m)+\zeta)}{\beta m(\zeta+\beta(1-\zeta))}$$

$$c_2=\frac{\zeta(m-1)-\beta(2\zeta m-\zeta-m)}{m(1-\beta)(\zeta+\beta(1-\zeta))}.$$

The terms u, v, p, θ, h are perturbations of the basic flow. In terms of normal modes

$$\mathbf{v}(x,y,t)=\{u(y),\mathrm{i}v(y)\}\exp(\mathrm{i}\alpha(x-ct));$$

$$p(x,y,t)=p(y)\exp(\mathrm{i}\alpha(x-ct));$$

$$\theta(x,y,t)=\theta(y)\exp(\mathrm{i}\alpha(x-ct));$$

$$h(x,t)=h\exp(\mathrm{i}\alpha(x-ct));$$

we get equations for $u(y)$, $v(y)$, $p(y)$, $\theta(y)$ and h:

$$\alpha u+v'=0; \tag{51}$$

$$\alpha(U-c)u+v\frac{\mathrm{d}U}{\mathrm{d}y}=-\alpha p+\frac{\mathrm{i}}{\mathbb{R}}(\alpha^2u-u''); \tag{52}$$

$$\alpha(U-c)v=p'+\frac{\mathrm{i}}{\mathbb{R}}(\alpha^2v-v''); \tag{53}$$

$$\alpha(U-c)\theta+v\frac{\mathrm{d}T}{\mathrm{d}y}=\frac{\mathrm{i}}{Pe}(\alpha^2\theta-\theta''); \tag{54}$$

together with boundary conditions:

$$u_1(r)=v_1(r)=\theta_1(r)=0; \tag{55}$$

$$u_2(0)=v_2(0)=\theta_2(0)=0. \tag{56}$$

After $p(y)$ and h have been eliminated, the jump conditions at the unperturbed interface $y=1$ can be expressed as

$$\alpha[u](U_1-\zeta U_2)+(v_1-\zeta v_2)\left[\frac{\mathrm{d}U}{\mathrm{d}y}\right]=\alpha c(1-\zeta)[u] \tag{57}$$

$$-\left(\frac{\mathrm{d}^2U_1}{\mathrm{d}y^2}-m\frac{\mathrm{d}^2U_2}{\mathrm{d}y^2}\right)u_1+\left(\frac{\mathrm{d}^2U_1}{\mathrm{d}y^2}-m\frac{\mathrm{d}^2U_2}{\mathrm{d}y^2}\right)u_2$$
$$+u_1'\left[\frac{\mathrm{d}U}{\mathrm{d}y}\right]-mu_2'\left[\frac{\mathrm{d}U}{\mathrm{d}y}\right]-\alpha v_1\left[\frac{\mathrm{d}U}{\mathrm{d}y}\right]$$
$$+\alpha mv_2\left[\frac{\mathrm{d}U}{\mathrm{d}y}\right]=0 \tag{58}$$

$$\frac{\mathrm{i}}{\alpha\mathbb{R}_1}u_1''\left[\frac{\mathrm{d}U}{\mathrm{d}y}\right]-\frac{\mathrm{i}\zeta}{\alpha\mathbb{R}_2}u_2''\left[\frac{\mathrm{d}U}{\mathrm{d}y}\right]-\left\{\left(\frac{\mathrm{i}\alpha}{\mathbb{R}_1}-U_1\right)\left[\frac{\mathrm{d}U}{\mathrm{d}y}\right]\right.$$
$$\left.-W\alpha^2+(1-\zeta)\tilde{G}\right\}u_1+\left\{\zeta\left(\frac{\mathrm{i}\alpha}{\mathbb{R}_2}-U_2\right)\left[\frac{\mathrm{d}U}{\mathrm{d}y}\right]\right.$$
$$\left.-W\alpha^2+(1-\zeta)\tilde{G}\right\}u_2+2\left[\frac{\mathrm{d}U}{\mathrm{d}y}\right]\left(\frac{\mathrm{i}}{\mathbb{R}_1}v_1'-\frac{\zeta\mathrm{i}}{\mathbb{R}_2}v_2'\right)$$
$$+\left(\frac{\mathrm{d}U_1}{\mathrm{d}y}\frac{v_1}{\alpha}-\zeta\frac{\mathrm{d}U_2}{\mathrm{d}y}\frac{v_2}{\alpha}\right)\left[\frac{\mathrm{d}U}{\mathrm{d}y}\right]=(u_1-\zeta u_2)c\left[\frac{\mathrm{d}U}{\mathrm{d}y}\right] \tag{59}$$

$$(\alpha u_2-\alpha u_1)U_1-v_1\left[\frac{\mathrm{d}U}{\mathrm{d}y}\right]-\frac{\mathrm{i}\Gamma}{\mathbb{R}_1}\theta_1'\left[\frac{\mathrm{d}U}{\mathrm{d}y}\right]$$
$$+\frac{\mathrm{i}\Gamma}{\mathbb{R}_1}\xi\theta_2'\left[\frac{\mathrm{d}U}{\mathrm{d}y}\right]=-\alpha(u_1-u_2)c. \quad (60)$$

The condition (44) that the temperature on the vapor side of the interface is at saturation can be written as

$$\Pi_1\left\{\mathrm{i}u_1''-(\mathrm{i}\alpha^2-\alpha\mathbb{R}_1U_1)u_1+\mathbb{R}_1v_1\frac{\mathrm{d}U_1}{\mathrm{d}y}\right\}$$
$$+\alpha\mathbb{R}_1\left(\theta_1-(T_1'+\Pi_1\tilde{G})\frac{[u]}{[U']}\right)=\alpha\Pi_1\mathbb{R}_1u_1c. \quad (61)$$

Similarly for water side

$$\Pi_2\left\{\mathrm{i}u_2''-(\mathrm{i}\alpha^2-\alpha\mathbb{R}_2U_2)u_2+\mathbb{R}_2v_2\frac{\mathrm{d}U_2}{\mathrm{d}y}\right\}$$
$$+\alpha\mathbb{R}_2\left(\theta_2-(T_2'+\Pi_2\tilde{G})\frac{[u]}{[U']}\right)=\alpha\Pi_2\mathbb{R}_2u_2c. \quad (62)$$

The continuity of temperature requires that

$$\left[\frac{\mathrm{d}U}{\mathrm{d}y}\right][\theta]-\left[\frac{\mathrm{d}T}{\mathrm{d}y}\right][u]=0. \quad (63)$$

Equations (61)–(63) cannot be enforced simultaneously. A choice must be made.

7. TWO DIFFERENT INTERFACIAL TEMPERATURE CONDITIONS

When (61) and (62) are adopted, the thermodynamic equilibrium is required at the interface, consequently the temperature continuity is not guaranteed. This closes the mathematical formation of the eigenvalue problem. We shall designate this problem as case I. Alternatively we can satisfy thermal equilibrium, namely (63) and require that one of the phases, say, the vapor phase be at saturation temperature. We designate this problem as case II.

These two different systems of equations differ only on one equation, although generally it should be expected that they will give different results, for the water–vapor cases, the differences are insignificant (Table 1). Hence, only the results of case II will be reported.

8. ENERGY ANALYSIS

The temperature or thermal energy equation is coupled to mechanical energy through the mechanism of phase change. In this case we may get two energy identities, one for mechanical energy designated with a subscript M and another for thermal energy designated with subscript T. The mechanical energy equation is obtained as follows. Suppose (u, v, θ) are the components of an eigenvector associated with the maximum growth rate of one of the problems satisfying the equations (51)–(61) and one of the equations of (62) or (63). To get the equation governing the evolution of the mechanical energy of the disturbance, we multiply (52) and (53) with u^* and v^*, the complex conjugates of u and v respectively, then integrate the sum of them over both the liquid and vapor regions

$$\langle\alpha(U-c)(u^2+v^2)\rangle=\frac{\mathrm{i}}{\mathbb{R}}\left\langle\alpha^2(u^2+v^2)\right.$$
$$\left.+\left(\frac{\mathrm{d}u}{\mathrm{d}y}\right)^2+\left(\frac{\mathrm{d}v}{\mathrm{d}y}\right)^2\right\rangle-\left\langle vu^*\frac{\mathrm{d}U}{\mathrm{d}y}\right\rangle+\left[\frac{\zeta\mathrm{i}}{\mathbb{R}}(u'u^*+v'v^*)\right]$$
$$-[\zeta pv^*] \quad (64)$$

where $\langle\ \rangle=\int_1+\int_2$, $u^2=uu^*$, $v^2=vv^*$ and we used the boundary and interfacial conditions to evaluate the integrals. Equation (64) is the energy balance for the small disturbance, and every term in it can be interpreted as some kind of energy. The imaginary part of the left hand side represents the growth of the energy of the disturbance and the right hand side may be split into three parts

$$\dot{E}_{\mathrm{M}}=I_{\mathrm{M}}-D_{\mathrm{M}}+B_{\mathrm{M}} \quad (65)$$

where

$$\dot{E}_{\mathrm{M}}=\alpha c_i\langle(u^2+v^2)\rangle$$
$$I_{\mathrm{M}}=\mathrm{Im}\left\langle vu^*\frac{\mathrm{d}U}{\mathrm{d}y}\right\rangle$$
$$D_{\mathrm{M}}=\frac{1}{\mathbb{R}}\left\langle\alpha^2(u^2+v^2)+\left(\frac{\mathrm{d}u}{\mathrm{d}y}\right)^2+\left(\frac{\mathrm{d}v}{\mathrm{d}y}\right)^2\right\rangle$$
$$B_{\mathrm{M}}=-\mathrm{Im}\left\{\left[\frac{\zeta\mathrm{i}}{\mathbb{R}}(u'u^*+v'v^*)\right]-[\zeta pv^*]\right\}.$$

We may transform the last term of B_{M} as follows:

Table 1. A comparison of the results from different choices of interfacial temperature conditions for typical parameters in the vertical case ($\omega = 90°$)

Parameters	Eigenvalue with the maximum growth rate Case I	 Case II
$\mathbb{R}_2 = 10$ $r = 1.5$ $\tau_2 = 0.97$	$\tilde{\alpha} = 0.0130$ $\tilde{\alpha}c = 1.040555, 0.152185\mathrm{i}$	$\tilde{\alpha} = 0.0131$ $\tilde{\alpha}c = 1.040536, 0.152197\mathrm{i}$
$\mathbb{R}_2 = 1000$ $r = 2.0$ $\tau_2 = 0.97$	$\tilde{\alpha} = 0.250$ $\tilde{\alpha}c = 0.266496, 0.0072672\mathrm{i}$	$\tilde{\alpha} = 0.249$ $\tilde{\alpha}c = 0.265408, 0.0072667\mathrm{i}$

$$[\zeta p v^*] = [\zeta p] v_2^* + [v^*] p_1 \tag{66}$$

where $[\zeta p]$ can be evaluated from the jump condition as

$$2\left[\frac{\zeta \mathrm{i}}{\mathbb{R}} v'\right] + (\alpha^2 W - (1-\zeta)\tilde{G}) \frac{[u]}{\left[\frac{\mathrm{d}U}{\mathrm{d}y}\right]}.$$

Then B_{M} can be further decomposed into four parts

$$B_{\mathrm{M}} = B_1 + Ph + B_2 + G \tag{67}$$

where

$$B_1 = \alpha^2 W \,\mathrm{Im}\left(\frac{[u] v_2^*}{\left[\frac{\mathrm{d}U}{\mathrm{d}y}\right]}\right)$$

$$Ph = \mathrm{Im}\,([v^*] p_1)$$

which arises from the change of phase:

$$B_2 = -\mathrm{Im}\left\{\left[\frac{\zeta \mathrm{i}}{\mathbb{R}} (u' u^* + v' v^*)\right] - 2\left[\frac{\zeta \mathrm{i}}{\mathbb{R}} v'\right] v_2^*\right\}$$

and

$$G = \tilde{G}(\zeta - 1)\,\mathrm{Im}\left(\frac{[u] v_2^*}{\left[\frac{\mathrm{d}U}{\mathrm{d}y}\right]}\right)$$

is the gravity term.

To get the equation governing the production of thermal energy we multiply equation (54) by θ^*, and integrate over both vapor and water regions

$$\langle \alpha (U-c) \theta^2 \rangle = \left\langle \frac{\mathrm{i}}{Pe}\left(\alpha^2 \theta^2 + \left(\frac{\mathrm{d}\theta}{\mathrm{d}y}\right)^2\right)\right\rangle - \left\langle v \theta^* \frac{\mathrm{d}T}{\mathrm{d}y}\right\rangle + \left[\frac{\mathrm{i}}{Pe} \theta' \theta^*\right]. \tag{68}$$

Equation (68) can be written as follows:

$$\dot{E}_{\mathrm{T}} = I_{\mathrm{T}} - D_{\mathrm{T}} + B_{\mathrm{T}} \tag{69}$$

where

$$\dot{E}_{\mathrm{T}} = \alpha c_i \langle \theta^2 \rangle$$

$$D_{\mathrm{T}} = \left\langle \frac{1}{Pe}\left(\alpha^2 \theta^2 + \left(\frac{\mathrm{d}\theta}{\mathrm{d}y}\right)^2\right)\right\rangle$$

$$I_{\mathrm{T}} = \mathrm{Im}\left\langle v \theta^* \frac{\mathrm{d}T}{\mathrm{d}y}\right\rangle$$

$$B_{\mathrm{T}} = -\mathrm{Re}\left(\left[\frac{1}{Pe} \theta' \theta^*\right]\right).$$

9. HORIZONTAL CASE

The horizontal case deserves special treatment because in this case gravity does not drive the flow. The basic state is motionless with a linear temperature profile and zero velocity. Since there is no prescribed velocity a different unit of velocity scale, $\mu_1 \zeta / \rho_1 H_0 m$ is used to make the equations dimensionless. The governing equations then become

$$v^{(4)} - 2\alpha^2 v'' + \alpha^4 v = \mathrm{i}\alpha c \mathbb{R} (\alpha^2 v - v'') \tag{70}$$

$$\theta'' - \alpha^2 \theta - \mathrm{i} Pe\, v T' = \mathrm{i}\alpha c\, Pe\, \theta. \tag{71}$$

The general solution of this system can be expressed as

$$v(y) = A\,\mathrm{e}^{\alpha y} + B\,\mathrm{e}^{-\alpha y} + C\,\mathrm{e}^{\hat{\alpha} y} + D\,\mathrm{e}^{-\hat{\alpha} y} \tag{72}$$

$$\theta(y) = E\,\mathrm{e}^{\hat{\gamma} y} + F\,\mathrm{e}^{-\hat{\gamma} y} + \tilde{\theta}(y) \tag{73}$$

where

$$\hat{\alpha}^2 = \alpha^2 - \mathrm{i}\alpha c \mathbb{R}, \quad \hat{\gamma}^2 = \alpha^2 - \mathrm{i}\alpha c\, Pe \quad \text{and}$$

$$\tilde{\theta}(y) = \frac{T'}{\alpha c}\left[A\,\mathrm{e}^{\alpha y} + B\,\mathrm{e}^{-\alpha y} + \frac{Pe}{Pe - \mathbb{R}} (C\,\mathrm{e}^{\hat{\alpha} y} + D\,\mathrm{e}^{-\hat{\alpha} y})\right].$$

Substituting (72) and (73) into the boundary and interface conditions, we are led to an eigenvalue problem for a 13 × 13 matrix acting on a vector whose 13 components are the coefficients A, B, C, D, E, F for both vapor and water phase and the interface position variable h. This system cannot be solved analytically. However for each set of parameters, we can find the eigenvalues numerically. There are three independent parameters H_0, r and τ_2. Figure 2 displays the neutral curve when H_0 and r are fixed, τ_2 varies. It shows that when $1 - \tau_2 = (T_s - T_-)/T_s$ increases at a fixed value of α the basic state becomes unstable. The case in which τ_1 and H_0 are fixed and r changes is represented in Fig. 3 which shows that the basic state is more stable when the vapor layer is thicker.

The energy analysis (Table 2) shows that the phase change term is the dominant destabilizing term in the energy balance, surface tension and gravity are

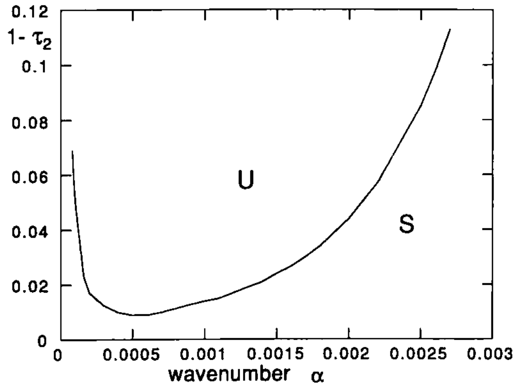

FIG. 2. Neutral curve when $H_0 = 0.001$ m, $r = 1.3$, $\tilde{G} = 0.00434$, $W = 48.4$, Γ, Π_1 and Π_2 change with $(1 - \tau_2)$, in the horizontal case.

Table 2. Energy analysis: $H_0 = 0.001$ m; $\tilde{G} = 0.00434$; $W = 48.4$; and $\tau_1 = 1.2$ (in the horizontal case)

$1-\tau_2$	$\tilde{\alpha}$	$\tilde{\alpha}c_i$	$\dot{E}_M$	B_1	Ph	B_2	G	$\dot{E}_T$	B_T	I_T-D_T
0.017	3.6e-4	1.6e-6	1.2e-3	−4.6e-9	1.00	2.0e-5	−5.5e-7	7.1e-6	1.0	−9.99e-1
0.070	5.2e-3	4.0e-3	3.8e-1	−1.0e-8	1.38	3.5e-6	−5.6e-4	1.0e-3	1.0	−9.99e-1

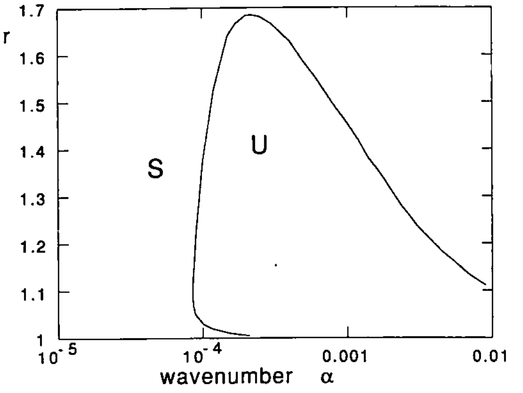

FIG. 3. Neutral curve when τ_2 (i.e. the temperature of the lower plane) is fixed (0.95): $H_0 = 0.001$ m; $\tilde{G} = 0.00434$; $W = 48.4$; $\Gamma = 0.231e^{-3}$; $\Pi_1 = 0.2e^{-4}$; $\Pi_2 = 0.0338$, in the horizontal case.

stabilizing but of small magnitude. The instability is solely caused by the change of phase.

10. VERTICAL CASE

In the general case an analytic form for the eigenfunction cannot be found and the finite element code of Hu and Joseph [17] was used to solve the eigenvalue problem. Figure 4 displays neutral curves in the vertical case when $r = 2.0$, $\tau_2 = 0.97$.

The neutral curve consists of two branches, each of which represents a different mechanism of instability. The energy analysis of this case (Table 3, where the viscous dissipation term D_M is used to normalize (65) and D_T is used to normalize (69)) shows that in region I the instability is caused by the interfacial friction B_2 together with the phase change term P when the Reynolds number is small. On the other hand the instability in region II arises from the Reynolds stress in the water layer. There is an overlapping region where two unstable modes exist, and a mode jump is observed at $\mathbb{R}_2$ about 1130.

When the vapor layer is thinner (Fig. 5, where $r = 1.5$, $\tau_2 = 0.97$), another instability appears in the long wave range. Energy analysis (Table 4) shows that this instability arises from phase change. We can also see that the maximum growth rate decreases as the Reynolds number increases and at $\mathbb{R}_2$ about 670, a mode jump occurs. The new mode of instability is associated with the Reynolds stress with phase change playing only a minor role.

Figure 6 displays the neutral curves when Reynolds number and r are fixed ($\mathbb{R}_2 = 1000$, $r = 1.5$) and τ_2 varies. Three different instabilities can be identified. For long waves there is an unstable mode associated with phase change, consistent with the results for the horizontal case, this mode goes unstable when $1-\tau_2 = (T_s - T_-)/T_s$ increases at a fixed α. However in this case the mode with the maximum growth rate is associated with the Reynolds stress, which is always unstable (right most region of Fig. 6), while the

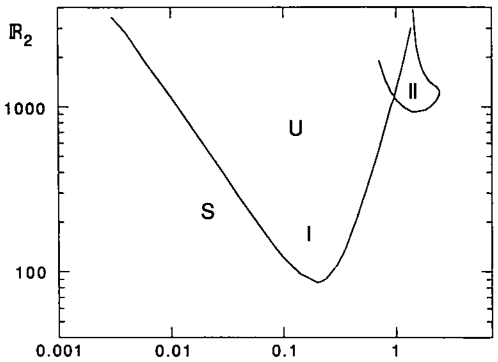

FIG. 4. Neutral curve when $r = 2.0$, $\tau_2 = 0.97$, in the vertical case.

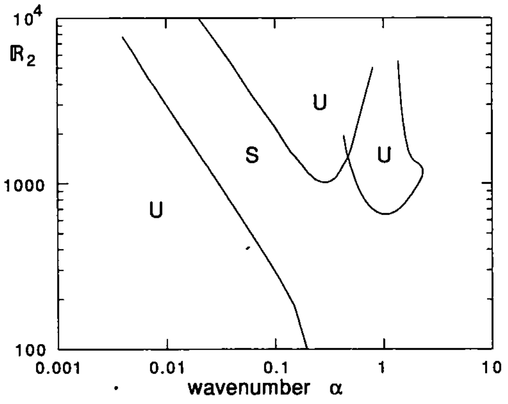

FIG. 5. Neutral curve for vertical case $\omega = 90°$, $r = 1.5$, $\tau_2 = 0.97$.

Table 3. Energy analysis for the mode of maximum growth rate in the vertical case: $r = 2.0$; $\tau_2 = 0.97$

$\mathbb{R}_2$	$\tilde{\alpha}$	$\tilde{\alpha}c_i$	$\dot{E}_M$	I_M-D_M	B_1	Ph	B_2	$\dot{E}_T$	B_T	I_T-D_T
86.7	0.198	1.40e-5	8.2e-4	−0.97	−2.1e-3	0.546	0.427	5.7e-6	0.72	−0.720
900	0.243	7.43e-3	1.2e-1	−0.98	−1.0e-1	0.072	1.130	3.2e-2	0.57	−0.537
1200	1.491	2.07e-2	2.16	3.26	−0.873	−0.002	−0.225	4.2e-2	0.71	−0.672
2000	1.176	3.50e-2	2.96	3.29	−0.314	0.002	−0.012	1.4e-1	0.72	−0.583

Table 4. Energy analysis for the mode of maximum growth rate in the vertical case (Fig. 5): $r = 1.5$; $\tau_2 = 0.97$

$\mathbb{R}_2$	$\tilde{\alpha}$	$\tilde{\alpha}c_i$	$\dot{E}_M$	$I_M - D_M$	B_1	Ph	B_2	$\dot{E}_T$	B_T	$I_T - D_T$
100	0.0060	1.52e-2	6.9e-2	−1.000	−6.8e-8	1.07	3.5e-3	1.7e-3	0.964	−0.962
300	0.0041	5.12e-3	6.9e-2	−0.999	−2.2e-8	1.06	7.9e-3	1.7e-3	0.964	−0.962
650	0.0031	2.28e-3	6.9e-2	−0.998	−9.4e-9	1.05	1.4e-2	1.7e-3	0.964	−0.962
700	1.041	5.67e-3	6.7e-1	1.09	−3.1e-1	0.11	−0.21	3.4e-3	1.24	−1.239
2000	1.065	8.11e-2	5.61	6.06	−4.8e-1	0.07	2.6e-2	0.121	1.40	−1.280

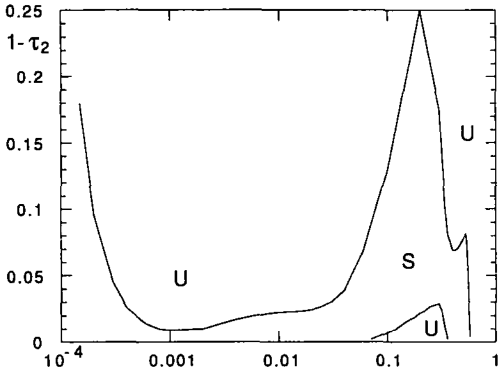

Fig. 6. Neutral curve when Reynolds number and r are fixed: $\mathbb{R}_2 = 1000$; $r = 1.5$, in the vertical case.

unstable region resulted from the interfacial friction is split into two parts.

Finally we briefly discuss the effect of the inclination angle ω. Notice that in our dimensionless form ω only appears in the gravity parameter, and from the energy analysis we see, as can be expected when the vapor lies above the water, that gravity is stabilizing, however when phase change is the dominant destabilizing factor, the influence of the gravity is almost negligible. In the inclined cases, there is an unstable mode caused by the interfacial friction, in these cases the gravity term is larger but not large enough to stabilize the flow (Table 5).

11. CONCLUSIONS

The problem of stability of fully developed flow of liquid and its vapor under gravity between heated and cold parallel plates was considered. The water and its vapor were assumed to be incompressible, but vaporization and condensation at the water–vapor interface could occur. In the fully developed case there is no phase change at the flat interface but the perturbation induces phase changes under the condition that the total volume of vapor and the total volume of liquid is conserved. Three kinds of instability can arise, an instability due to the Reynolds stress at higher Reynolds numbers, an instability due to interfacial friction which is associated with the viscosity difference and will arise even in the absence of phase change and a strong phase change instability at the interface which can occur even between horizontal plates heated above with no basic motion. All these instabilities arise as overstability so that Hopf bifurcation into periodic solutions is expected. Analysis of bifurcations, presently underway, should reveal whether periodic solutions are stable, can be observed, and decide when the waves propagate.

Table 5. Critical points in the case when $r = 2.0$, $\tau_2 = 0.97$ with different inclination angles

ω	Critical Reynolds number, $\tilde{R}_2$	Critical wave number, $\tilde{\alpha}$	Wavespeed, Re($\tilde{\alpha}c$)
90°	86.7	0.198	0.305
30°	93.4	0.186	0.296
10°	106.1	0.169	0.272

The issue of temperature conditions at a phase change interface is an important one. The problem arises whenever a pressure jump across the interface is allowed even when the phases are solid or liquid. If the pressures on each side of the interface are different and the temperature is at saturation, then there must be a discontinuity of temperature. Different choices of thermal interface conditions are possible. Although in special cases treated here, water and its vapor, the stability results do not depend sensitively on the choice of conditions of the temperature at the interface we do not expect weak dependence in general especially when the dependence of the saturation temperature on the pressure is not small. The choice of temperature conditions at a phase change boundary goes beyond continuum thermomechanics and appears to require some form of molecular theory.

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